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Generalized Pandita Numbers

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Abstract. In this paper, we introduce and investigate the generalized Pandita sequences and we deal with, in detail, two special cases, namely, Pandita and Pandita-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Pandita, Pandita-Lucas and Narayana, Narayana-Lucas numbers.

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1. Introduction

Narayana sequence$\{N_n\}_{n\geq 0}$ (OEIS: A000930, [6]) and Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ (OEIS: A001609, [6]) are defined, respectively, by the third-order recurrence relations

$$N_{n+3} = N_{n+2} + N_n, \qquad N_0 = 0, N_1 = 1, N_2 = 1, \tag{1.1}$$

$$U_{n+3} = U_{n+2} + U_n, \qquad U_0 = 3, U_1 = 1, U_2 = 1. \tag{1.2}$$

The sequences $\{N_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$N_{-n} = -N_{-(n-2)} + N_{-(n-3)},$$
$$U_{-n} = -U_{-(n-2)} + U_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1)- (1.2) hold for all integer n. For more information on generalized Narayana numbers, see Soykan [13].

Now, we define two sequences related to Narayana and Narayana-Lucas numbers. Pandita and Pandita-Lucas numbers are defined as

$$P_n = P_{n-1} + P_{n-3} + 1, \quad \text{with } P_0 = 0, P_1 = 1, P_2 = 2, \qquad n \geq 3,$$

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and

$$S_n = S_{n-1} + S_{n-3} - 1, \qquad \text{with} \ \ S_0 = 4, S_1 = 2, S_2 = 2, \qquad n \geq 3,$$

respectively.

The first few values of Pandita and Pandita-Lucas numbers are

$$0, 1, 2, 3, 5, 8, 12, 18, 27, 40, 59, 87, 128, 188, ...$$

and

$$4, 2, 2, 5, 6, 7, 11, 16, 32, 47, 68, 99, 145, ...$$

respectively.

The sequences $\{P_n\}$ and $\{S_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{aligned} P_n &= 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \qquad n \geq 4, \\ S_n &= 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \qquad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \qquad n \geq 4. \end{aligned}$$

There are close relations between Pandita, Pandita-Lucas and Narayana, Narayana-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} P_n &= N_{n+2} + N_n - 1 = N_{n+3} - 1, \\ 31P_n &= 13U_{n+2} + 6U_{n+1} + 4U_n - 31, \\ S_n &= 3N_{n+1} - 2N_n + 1, \\ S_n &= U_n + 1, \end{aligned}$$

and

$$\begin{aligned} N_{n+3} &= P_{n+3} - P_{n+2}, \\ 31N_n &= 9S_{n+2} - 3S_{n+1} - 2S_n - 4, \\ U_n &= 2P_{n+2} + P_{n+1} - 5P_n - 2, \\ U_n &= S_{n+3} - S_{n+2}. \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Pandita, Pandita-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \qquad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1.3}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1,3,4,5,8,10,11,14,15]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence (1.3) holds for all integers n.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{1.4}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= r, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

THEOREM 1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) For all integers n, Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{1.5}$$

where

$$\begin{aligned} p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (1.4) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (1.4) for non-negative integers is valid for all integers n (see [2]).

Next, we consider two special cases of the generalized (r,s,t,u) sequence $\{W_n\}$ which we call them (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas sequences. (r,s,t,u)-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and (r,s,t,u)-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n, \qquad (1.6)\\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, \\ H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n, \qquad (1.7)\\ H_0 &= 4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.6) and (1.7) hold for all integers n.

For all integers n, (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers (using initial conditions in (1.6) or (1.7)) can be expressed using Binet's formulas as in the following corollary.

COROLLARY 2. *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta$) Binet's formula of (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers are*

$$G_n=\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

and

$$H_n=\alpha^n+\beta^n+\gamma^n+\delta^n,$$

respectively.

Proof. Take $W_n=G_n$ and $W_n=H_n$ in Theorem 1, respectively. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty}W_nz^n$ of the sequence W_n.

LEMMA 3. *Suppose that $f_{W_n}(z)=\sum_{n=0}^{\infty}W_nz^n$ is the ordinary generating function of the generalized (r,s,t,u) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty}W_nz^n$ is given by*

$$\sum_{n=0}^{\infty}W_nz^n=\frac{W_0+(W_1-rW_0)z+(W_2-rW_1-sW_0)z^2+(W_3-rW_2-sW_1-tW_0)z^3}{1-rz-sz^2-tz^3-uz^4}. \qquad (1.8)$$

Proof. For a proof, see Soykan [8, Lemma 1]. □

The following theorem presents Simson's formula of generalized (r,s,t,u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

Theorem 4 (Simson's Formula of Generalized (r,s,t,u) Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_{n} \\ W_{n+2} & W_{n+1} & W_{n} & W_{n-1} \\ W_{n+1} & W_{n} & W_{n-1} & W_{n-2} \\ W_{n} & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_{3} & W_{2} & W_{1} & W_{0} \\ W_{2} & W_{1} & W_{0} & W_{-1} \\ W_{1} & W_{0} & W_{-1} & W_{-2} \\ W_{0} & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{1.9}$$

Proof. (1.9) is given in Soykan [7]. □

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 5. *For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r,s,t,u)-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n}+6H_nW_{2n}-3H_n^2W_n+3H_{2n}W_n+W_0H_n^3+2W_0H_{3n}-3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n}-H_nW_{2n}+\frac{1}{2}(H_n^2-H_{2n})W_n-\frac{1}{6}(H_n^3+2H_{3n}-3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [9, Theorem 1.]. □

Using Theorem 5, we have the following corollary, see Soykan [9, Corollary 4].

Corollary 6. *For $n \in \mathbb{Z}$, we have*

(a): $2(-u)^{n+4}G_{-n} = -(3ru^2+t^3-3stu)^2G_n^3-(2su-t^2)^2G_{n+3}^2G_n-(-rt^2-tu+2rsu)^2G_{n+2}^2G_n$ $-(-st^2+2s^2u+4u^2+rtu)^2G_{n+1}^2G_n+2(3ru^2+t^3-3stu)((-2su+t^2)G_{n+3}+(-rt^2-tu+2rsu)G_{n+2}+$ $(-st^2+2s^2u+4u^2+rtu)G_{n+1})G_n^2+2(2su-t^2)(-rt^2-tu+2rsu)G_{n+3}G_{n+2}G_n+2(2su-t^2)(-st^2+$ $2s^2u+4u^2+rtu)G_{n+3}G_{n+1}G_n-2(-st^2+2s^2u+4u^2+rtu)(-rt^2-tu+2rsu)G_{n+2}G_{n+1}G_n-2G_{3n}u^4+$ $u^2(-2su+t^2)G_{2n+3}G_n+u^2(-rt^2-tu+2rsu)G_{2n+2}G_n+u^2(-st^2+2s^2u+4u^2+rtu)G_{2n+1}G_n-$ $2u^2(2su-t^2)G_{2n}G_{n+3}+2u^2(-rt^2-tu+2rsu)G_{2n}G_{n+2}+2u^2(-st^2+2s^2u+4u^2+rtu)G_{2n}G_{n+1}-$ $3u^2(3ru^2+t^3-3stu)G_{2n}G_n$.

(b): $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right)$.

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n}+6H_nG_{2n}-3H_n^2G_n+3H_{2n}G_n), \tag{1.10}$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right), \tag{1.11}$$

respectively.

If we define the square matrix M of order 4 as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

THEOREM 7. *For all integers m, n, we have*

(a): $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b): $D_1 A^n = A^n D_1$.

(c): $D_{n+m} = D_n B_m = B_m D_n$.

Proof. For the proof, see Soykan [8, Theorem 19]. □

THEOREM 8. *For all integers m, n, we have*

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \tag{1.12}$$

Proof. For the proof, see Soykan [8, Theorem 20]. □

In the next sections, we present new results.

2. Generalized Pandita Sequence

In this paper, we consider the case $r = 2, s = -1, t = 1, u = -1$. A generalized Pandita sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = W_{-(n-1)} - W_{-(n-2)} + 2W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integers n.

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 + z^2 - z + 1 = (z^3 - z^2 - 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 2, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned}
\alpha + \beta + \gamma &= 1, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The first few generalized Pandita numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Pandita numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_0 - W_1 + 2W_2 - W_3$
2	W_2	$W_1 + W_2 - W_3$
3	W_3	$W_0 + W_1 - W_2$
4	$W_1 - W_0 - W_2 + 2W_3$	$2W_0 - 2W_1 + 2W_2 - W_3$
5	$W_1 - 2W_0 - W_2 + 3W_3$	$3W_2 - 2W_3$
6	$W_1 - 3W_0 - 2W_2 + 5W_3$	$3W_1 - 2W_2$
7	$2W_1 - 5W_0 - 4W_2 + 8W_3$	$3W_0 - 2W_1$
8	$3W_1 - 8W_0 - 6W_2 + 12W_3$	$W_0 - 3W_1 + 6W_2 - 3W_3$
9	$4W_1 - 12W_0 - 9W_2 + 18W_3$	$5W_1 - 2W_0 - W_2 - W_3$
10	$6W_1 - 18W_0 - 14W_2 + 27W_3$	$3W_0 + W_1 - 5W_2 + 2W_3$
11	$9W_1 - 27W_0 - 21W_2 + 40W_3$	$4W_0 - 8W_1 + 8W_2 - 3W_3$
12	$13W_1 - 40W_0 - 31W_2 + 59W_3$	$4W_1 - 4W_0 + 5W_2 - 4W_3$
13	$19W_1 - 59W_0 - 46W_2 + 87W_3$	$9W_1 - 12W_2 + 4W_3$

Note that the sequences $\{P_n\}$ and $\{S_n\}$ which are defined in the section Introduction, are the special cases of the generalized Pandita sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$, in this section as well. Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita-Lucas sequence $\{S_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$
\begin{aligned}
P_n &= 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \qquad n \geq 4, \\
S_n &= 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \qquad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \qquad n \geq 4.
\end{aligned}
$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$
\begin{aligned}
P_{-n} &= P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}, \\
S_{-n} &= S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)},
\end{aligned}
$$

for $n = 1, 2, 3, ...$ respectively.

Next, we present the first few values of the Pandita and Pandita-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	3	5	8	12	18	27	40	59	87	128	188
P_{-n}	0	0	0	-1	-1	0	-1	-2	0	0	-3	-1	2	-3
S_n	4	2	2	5	6	7	11	16	22	32	47	68	99	145
S_{-n}	4	1	-1	4	3	-4	2	8	-5	-5	14	1	-18	14

(1.5) can be used to obtain the Binet formula of generalized Pandita numbers. Binet's formula of generalized Pandita numbers can be given as follows:

THEOREM 9. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of generalized Pandita numbers is*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+1)W_1 - W_0)\alpha^n}{3\alpha - 2} \\ &+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+1)W_1 - W_0)\beta^n}{3\beta - 2} \\ &+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+1)W_1 - W_0)\gamma^n}{3\gamma - 2} \\ &- W_3 + W_2 + W_0. \end{aligned}$$

Pandita and Pandita-Lucas numbers can be expressed using Binet's formulas as follows.

COROLLARY 10. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of Pandita and Pandita-Lucas numbers are*

$$P_n = \frac{\alpha^{n+3}}{3\alpha-2} + \frac{\beta^{n+3}}{3\beta-2} + \frac{\gamma^{n+3}}{3\gamma-2} - 1,$$

and

$$S_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

Note that Binet's formulas of Narayana and Narayana-Lucas numbers, respectively, are

$$\begin{aligned} N_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ U_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

see, Soykan [13] for more details.

So, by using Binet's formulas of Pandita, Pandita-Lucas and Narayana, Narayana-Lucas numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

LEMMA 11. *For all integers n, the following equalities (identities) are true:*

(a):

- $N_{n+3} = P_{n+3} - P_{n+2}$.
- $N_n = P_{n+3} - 2P_{n+2} + P_{n+1}$.
- $P_{n+4} = 4N_{n+2} + 2N_{n+1} + 3N_n - 1$.
- $P_n = N_{n+2} + N_n - 1 = N_{n+3} - 1$.
- $N_n = -P_{n+2} + P_{n+1} + P_n + 1$.
- $N_{n+1} = P_{n+1} - P_n$.

(b):

- $31N_{n+3} = 23S_{n+3} - 10S_{n+2} + 6S_{n+1} - 19S_n$.
- $31N_n = 4S_{n+3} + 5S_{n+2} - 3S_{n+1} - 6S_n$.
- $S_{n+4} = 4N_{n+2} + N_{n+1} + N_n + 1$.
- $S_n = 3N_{n+1} - 2N_n + 1$.
- $31N_n = 9S_{n+2} - 3S_{n+1} - 2S_n - 4$.

(c):

- $U_{n+3} = P_{n+3} - P_{n+2} + 3P_{n+1} - 3P_n$.
- $U_n = -2P_{n+3} + 4P_{n+2} + P_{n+1} - 3P_n$.
- $31P_{n+4} = 55U_{n+2} + 23U_{n+1} + 36U_n - 31$.
- $31P_n = 13U_{n+2} + 6U_{n+1} + 4U_n - 31$.
- $U_n = 2P_{n+2} + P_{n+1} - 5P_n - 2$.
- $13P_{n+1} - 19P_n = -2U_{n+1} + 3U_n + 6$.

(d):

- $U_{n+3} = 2S_{n+3} - S_{n+2} - S_n$.
- $U_n = S_{n+3} - S_{n+2}$.
- $S_{n+4} = U_{n+2} + U_{n+1} + U_n + 1$.
- $S_n = U_n + 1$.
- $U_n = S_n - 1$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

LEMMA 12. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Pandita sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 2W_0)z + (W_2 - 2W_1 + W_0)z^2 + (W_3 - 2W_2 + W_1 - W_0)z^3}{1 - 2z + z^2 - z^3 + z^4}.$$

Proof. Take $r = 2, s = -1, t = 1, u = -1$ in Lemma 3.

The previous lemma gives the following results as particular examples.

COROLLARY 13. *Generating functions of Pandita and Pandita-Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} P_n z^n &= \frac{z}{1 - 2z + z^2 - z^3 + z^4}, \\ \sum_{n=0}^{\infty} S_n z^n &= \frac{4 - 6z + 2z^2 - z^3}{1 - 2z + z^2 - z^3 + z^4}, \end{aligned}$$

respectively.

3. Simson Formulas

Now, we present Simson's formula of generalized Pandita numbers.

THEOREM 14 (Simson's Formula of Generalized Pandita Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (W_0 + W_2 - W_3)(-W_3^3 + 3W_2^3 - W_1^3 + W_0^3 + (5W_2 - 2W_1)W_3^2 +$$
$$(4W_0 - 5W_1 - 8W_3)W_2^2 + (4W_0 + 4W_2 - 5W_3)W_1^2 + (W_2 - 3W_1 - W_3)W_0^2 + 9W_1W_2W_3 - 3W_0W_2W_3 + 5W_0W_1W_3 - 7W_0W_1W_2).$$

Proof. Take $r = 2, s = -1, t = 1, u = -1$ in Theorem 4. □

The previous theorem gives the following results as particular examples.

COROLLARY 15. *For all integers n, the Simson's formulas of Pandita and Pandita-Lucas numbers are given as*

$$\begin{vmatrix} P_{n+3} & P_{n+2} & P_{n+1} & P_n \\ P_{n+2} & P_{n+1} & P_n & P_{n-1} \\ P_{n+1} & P_n & P_{n-1} & P_{n-2} \\ P_n & P_{n-1} & P_{n-2} & P_{n-3} \end{vmatrix} = 1$$

$$\begin{vmatrix} S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+2} & S_{n+1} & S_n & S_{n-1} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} \end{vmatrix} = -31$$

respectively.

4. Some Identities

In this section, we obtain some identities of Pandita and Pandita-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{P_n\}$.

LEMMA 16. *The following equalities are true:*

- **(a):** $W_n = (2W_0 - 2W_1 + 2W_2 - W_3)P_{n+5} + (5W_1 - 3W_0 - 5W_2 + 2W_3)P_{n+4} + (5W_2 - 3W_1 - 2W_3)P_{n+3} + (2W_3 - 3W_2)P_{n+2}$.
- **(b):** $W_n = (W_0 + W_1 - W_2)P_{n+4} + (3W_2 - W_1 - 2W_0 - W_3)P_{n+3} + (2W_0 - 2W_1 - W_2 + W_3)P_{n+2} + (2W_1 - 2W_0 - 2W_2 + W_3)P_{n+1}$.
- **(c):** $W_n = (W_1 + W_2 - W_3)P_{n+3} + (W_0 - 3W_1 + W_3)P_{n+2} + (3W_1 - W_0 - 3W_2 + W_3)P_{n+1} + (W_2 - W_1 - W_0)P_n$.
- **(d):** $W_n = (W_0 - W_1 + 2W_2 - W_3)P_{n+2} + (2W_1 - W_0 - 4W_2 + 2W_3)P_{n+1} + (2W_2 - W_0 - W_3)P_n + (W_3 - W_2 - W_1)P_{n-1}$.
- **(e):** $W_n = W_0P_{n+1} + (W_1 - 2W_0)P_n + (W_0 - 2W_1 + W_2)P_{n-1} + (W_1 - W_0 - 2W_2 + W_3)P_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times P_{n+5} + b \times P_{n+4} + c \times P_{n+3} + d \times P_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times P_5 + b \times P_4 + c \times P_3 + d \times P_2 \\ W_1 &= a \times P_6 + b \times P_5 + c \times P_4 + d \times P_3 \\ W_2 &= a \times P_7 + b \times P_6 + c \times P_5 + d \times P_4 \\ W_3 &= a \times P_8 + b \times P_7 + c \times P_6 + d \times P_5 \end{aligned}$$

we find that $a = 2W_0 - 2W_1 + 2W_2 - W_3, b = 5W_1 - 3W_0 - 5W_2 + 2W_3, c = 5W_2 - 3W_1 - 2W_3, d = 2W_3 - 3W_2$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{S_n\}$.

Lemma 17. *The following equalities are true:*

- **(a):** $31W_n = -(40W_0 - 11W_1 + 39W_2 - 37W_3)S_{n+5} + (43W_0 - 25W_1 + 52W_2 - 39W_3)S_{n+4} - (W_0 - 15W_1 + 25W_2 - 11W_3)S_{n+3} + (29W_0 - W_1 + 43W_2 - 40W_3)S_{n+2}$.
- **(b):** $31W_n = -(37W_0 + 3W_1 + 26W_2 - 35W_3)S_{n+4} + (39W_0 + 4W_1 + 14W_2 - 26W_3)S_{n+3} - (11W_0 - 10W_1 - 4W_2 + 3W_3)S_{n+2} + (40W_0 - 11W_1 + 39W_2 - 37W_3)S_{n+1}$.
- **(c):** $31W_n = -(35W_0 + 2W_1 + 38W_2 - 44W_3)S_{n+3} + (26W_0 + 13W_1 + 30W_2 - 38W_3)S_{n+2} + (3W_0 - 14W_1 + 13W_2 - 2W_3)S_{n+1} + (37W_0 + 3W_1 + 26W_2 - 35W_3)S_n$.
- **(d):** $31W_n = -(44W_0 - 9W_1 + 46W_2 - 50W_3)S_{n+2} + (38W_0 - 12W_1 + 51W_2 - 46W_3)S_{n+1} + (2W_0 + W_1 - 12W_2 + 9W_3)S_n + (35W_0 + 2W_1 + 38W_2 - 44W_3)S_{n-1}$.
- **(e):** $31W_n = -(50W_0 - 6W_1 + 41W_2 - 54W_3)S_{n+1} + (46W_0 - 8W_1 + 34W_2 - 41W_3)S_n - (9W_0 - 11W_1 + 8W_2 - 6W_3)S_{n-1} + (44W_0 - 9W_1 + 46W_2 - 50W_3)S_{n-2}$.

Now, we give a few basic relations between $\{P_n\}$ and $\{S_n\}$.

Lemma 18. *The following equalities are true:*

$$\begin{aligned} 31P_n &= 44S_{n+5} - 38S_{n+4} - 2S_{n+3} - 35S_{n+2}, \\ 31P_n &= 50S_{n+4} - 46S_{n+3} + 9S_{n+2} - 44S_{n+1}, \\ 31P_n &= 54S_{n+3} - 41S_{n+2} + 6S_{n+1} - 50S_n, \\ 31P_n &= 67S_{n+2} - 48S_{n+1} + 4S_n - 54S_{n-1}, \\ 31P_n &= 86S_{n+1} - 63S_n + 13S_{n-1} - 67S_{n-2}, \end{aligned}$$

and

$$\begin{aligned} S_n &= 3P_{n+5} - 2P_{n+4} - 6P_{n+3} + 4P_{n+2}, \\ S_n &= 4P_{n+4} - 9P_{n+3} + 7P_{n+2} - 3P_{n+1}, \\ S_n &= -P_{n+3} + 3P_{n+2} + P_{n+1} - 4P_n, \\ S_n &= P_{n+2} + 2P_{n+1} - 5P_n + P_{n-1}, \\ S_n &= 4P_{n+1} - 6P_n + 2P_{n-1} - P_{n-2}. \end{aligned}$$

5. Relations Between Special Numbers

In this section, we present identities on Pandita, Pandita-Lucas numbers and Narayana, Narayana-Lucas numbers. We know that

$$\begin{aligned} P_n &= N_{n+2} + N_n - 1, \\ S_n &= U_n + 1, \end{aligned}$$

Note also that from Lemma 16 and Lemma 17, we have the formulas of W_n as

$$\begin{aligned} W_n &= (W_1 + W_2 - W_3)P_{n+3} + (W_0 - 3W_1 + W_3)P_{n+2} \\ &\quad +(3W_1 - W_0 - 3W_2 + W_3)P_{n+1} + (W_2 - W_1 - W_0)P_n, \\ 31W_n &= -(35W_0 + 2W_1 + 38W_2 - 44W_3)S_{n+3} + (26W_0 + 13W_1 + 30W_2 - 38W_3)S_{n+2} \\ &\quad +(3W_0 - 14W_1 + 13W_2 - 2W_3)S_{n+1} + (37W_0 + 3W_1 + 26W_2 - 35W_3)S_n. \end{aligned}$$

Using the above identities, we obtain the relation of generalized Pandita numbers and Narayana, Narayana-Lucas numbers in the following forms:

LEMMA 19. *For all integers n, we have the following identities:*

(a): $W_n = (W_2 - W_1)N_{n+2} + (W_3 - 2W_2 + W_1)N_{n+1} + (W_1 - W_0)N_n - W_3 + W_2 + W_0.$

(b): $31W_n = (6W_3 - 8W_2 + 11W_1 - 9W_0)U_{n+2} + (-2W_3 + 13W_2 - 14W_1 + 3W_0)U_{n+1} + (9W_3 - 12W_2 + W_1 + 2W_0)U_n - 31W_3 + 31W_2 + 31W_0.$

6. On the Recurrence Properties of Generalized Pandita Sequence

Taking $r = 2, s = -1, t = 1, u = -1$ in Theorem 5, we obtain the following Proposition.

PROPOSITION 20. *For $n \in \mathbb{Z}$, generalized Pandita numbers (the case $r = 2, s = -1, t = 1, u = -1$) have the following identity:*

$$W_{-n} = \frac{1}{6}(-6W_{3n} + 6S_nW_{2n} - 3S_n^2W_n + 3S_{2n}W_n + W_0S_n^3 + 2W_0S_{3n} - 3W_0S_nS_{2n}).$$

From the above Proposition 20 (or by taking $G_n = P_n$ and $H_n = S_n$ in (1.10) and (1.11) respectively), we have the following corollary which gives the connection between the special cases of generalized Pandita sequence at the positive index and the negative index: for Pandita and Pandita-Lucas and Pandita numbers: take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3$ and take $W_n = S_n$ with $S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5$, respectively. Note that in this case $H_n = S_n$.

Corollary 21. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *Pandita sequence:*

$$P_{-n} = \frac{1}{2}(-2P_{3n} + 2S_n P_{2n} - S_n^2 P_n + S_{2n} P_n).$$

(b): *Pandita-Lucas sequence:*

$$S_{-n} = \frac{1}{6}(-6S_{3n} + 6S_n S_{2n} - 3S_n^2 S_n + 3S_{2n} S_n + 4S_n^3 + 8S_{3n} - 12S_n S_{2n}).$$

We can also present the formulas of P_{-n} and S_{-n} in the following forms.

Corollary 22. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): $P_{-n} = \frac{1}{2}(-2P_{3n} + 2(-P_{n+3} + 3P_{n+2} + P_{n+1} - 4P_n)P_{2n} - (-P_{n+3} + 3P_{n+2} + P_{n+1} - 4P_n)^2 P_n + (-P_{2n+3} + 3P_{2n+2} + P_{2n+1} - 4P_{2n})P_n)$.

(b): $P_{-n} = 2N_n^2 + 2N_{n-2}^2 - 3N_{n+1}N_n - 3N_{n-1}N_{n-2} + N_{2n} + N_{2n-4} - 1$.

(c): $S_{-n} = \frac{1}{2}(U_n^2 - U_{2n} + 2)$.

Proof.

(a): By using the identity $S_n = -P_{n+3} + 3P_{n+2} + P_{n+1} - 4P_n$ and Corollary 21, (or by using Corollary 6 (a)), we get (a).

(b): Since $P_n = N_{n+2} + N_n - 1$ and $N_{-n} = 2N_n^2 + N_{2n} - 3N_{n+1}N_n$ (see, for example Soykan [12]), we get (b).

(c): Since $S_n = U_n + 1$ and $U_{-n} = \frac{1}{2}(U_n^2 - U_{2n})$ (see, for example Soykan [12]), we obtain (c). □

7. Sum Formulas

The following Corollary gives sum formulas of Narayana and Narayana-Lucas numbers.

Corollary 23. *For $n \geq 0$, Narayana and Narayana-Lucas numbers have the following properties:*

(a):

- **(i):** $\sum_{k=0}^{n} N_k = N_{n+3} - 1$.
- **(ii):** $\sum_{k=0}^{n} N_{2k} = \frac{1}{3}(N_{2n+2} + N_{2n+1} + 2N_{2n} - 2)$.
- **(iii):** $\sum_{k=0}^{n} N_{2k+1} = \frac{1}{3}(2N_{2n+2} + 2N_{2n+1} + N_{2n} - 1)$.

(b):

- **(i):** $\sum_{k=0}^{n} U_k = U_{n+3} - 1$.

(ii): $\sum_{k=0}^{n} U_{2k} = \frac{1}{3}(U_{2n+2} + U_{2n+1} + 2U_{2n} + 1)$.

(iii): $\sum_{k=0}^{n} U_{2k+1} = \frac{1}{3}(2U_{2n+2} + 2U_{2n+1} + U_{2n} - 4)$.

Proof. It is given in Soykan [13, Corollary 6.1 and Corollary 6.2]. □

The following Corollary presents sum formulas of Pandita and Pandita-Lucas numbers.

Corollary 24. *For $n \geq 0$, Pandita and Pandita-Lucas numbers have the following properties:*

(a):

(i): $\sum_{k=0}^{n} P_k = 3N_{n+2} + N_{n+1} + 2N_n - n - 4$.

(ii): $\sum_{k=0}^{n} P_{2k} = \frac{1}{3}(5N_{2n+2} + 2N_{2n+1} + 4N_{2n} - 3n - 7)$.

(iii): $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{3}(7N_{2n+2} + 4N_{2n+1} + 5N_{2n} - 3n - 8)$.

(b):

(i): $\sum_{k=0}^{n} S_k = U_{n+2} + U_n + n$.

(ii): $\sum_{k=0}^{n} S_{2k} = \frac{1}{3}(U_{2n+2} + 2U_{2n} + U_{2n+1} + 3n + 4)$.

(iii): $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{3}(2U_{2n+2} + 2U_{2n+1} + U_{2n} + 3n - 1)$.

Proof. The proof follows from Corollary 23 and the identities

$$\begin{aligned} P_n &= N_{n+2} + N_n - 1, \\ S_n &= U_n + 1. \ \square \end{aligned}$$

8. Matrices and Identities Related With Generalized Pandita Numbers

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & -W_n + W_{n-1} - W_{n-2} & W_n - W_{n-1} & -W_n \\ W_n & -W_{n-1} + W_{n-2} - W_{n-3} & W_{n-1} - W_{n-2} & -W_{n-1} \\ W_{n-1} & -W_{n-2} + W_{n-3} - W_{n-4} & W_{n-2} - W_{n-3} & -W_{n-2} \\ W_{n-2} & -W_{n-3} + W_{n-4} - W_{n-5} & W_{n-3} - W_{n-4} & -W_{n-3} \end{pmatrix}$$

then we get the following Theorem.

THEOREM 25. *For all integers m, n, we have*

(a): $B_n = A^n$, *i.e.,*

$$A^n = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}.$$

(b): $D_1 A^n = A^n D_1$.

(c): $D_{n+m} = D_n B_m = B_m D_n$.

Proof. Take $r = 2, s = -1, t = 1, u = -1$ in Theorem 7. □

Using the above last Theorem and the identity

$$\begin{aligned} P_n &= N_{n+2} + N_n - 1 \\ &= N_{n+3} - 1, \end{aligned}$$

we obtain the following formula for Narayana numbers.

COROLLARY 26. *For all integers n, we have the following formula for Narayana numbers.*

$$A^n = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} N_{n+4} - 1 & -N_{n+4} + N_{n+2} + 1 & N_{n+3} - N_{n+2} & -N_{n+3} + 1 \\ N_{n+3} - 1 & -N_{n+3} + N_{n+1} + 1 & N_{n+2} - N_{n+1} & -N_{n+2} + 1 \\ N_{n+2} - 1 & -N_{n+2} + N_n + 1 & N_{n+1} - N_n & -N_{n+1} + 1 \\ N_{n+1} - 1 & -N_{n+1} + N_{n-1} + 1 & N_n - N_{n-1} & -N_n + 1 \end{pmatrix}.$$

Next, we present an identity for W_{n+m}.

THEOREM 27. *For all integers m, n, we have*

$$W_{n+m} = W_n P_{m+1} + W_{n-1}(-P_m + P_{m-1} - P_{m-2}) + W_{n-2}(P_m - P_{m-1}) - W_{n-3} P_m.$$

Proof. Take $r = 2, s = -1, t = 1, u = -1$ in Theorem 8. □

As particular cases of the above theorem, we give identities for P_{n+m} and S_{n+m}.

COROLLARY 28. *For all integers m, n, we have*

$$\begin{aligned} P_{n+m} &= P_n P_{m+1} + P_{n-1}(-P_m + P_{m-1} - P_{m-2}) + P_{n-2}(P_m - P_{m-1}) - P_{n-3} P_m, \\ S_{n+m} &= S_n P_{m+1} + S_{n-1}(-P_m + P_{m-1} - P_{m-2}) + S_{n-2}(P_m - P_{m-1}) - S_{n-3} P_m. \end{aligned}$$

Taking $m = n$ in the last corollary, we obtain the following identities:

$$\begin{aligned} P_{2n} &= P_n P_{n+1} + P_{n-1}(-P_n + P_{n-1} - P_{n-2}) + P_{n-2}(P_n - P_{n-1}) - P_{n-3} P_n, \\ S_{2n} &= S_n P_{n+1} + S_{n-1}(-P_n + P_{n-1} - P_{n-2}) + S_{n-2}(P_n - P_{n-1}) - S_{n-3} P_n. \end{aligned}$$

References

[1] G. S. Hathiwala, D. V. Shah, Binet–Type Formula For The Sequence of Tetranacci Numbers by Alternate Methods, Mathematical Journal of Interdisciplinary Sciences 6(1) (2017), 37–48.

[2] Howard F.T., Saidak, F., Zhou's Theory of Constructing Identities, Congress Numer. 200 (2010), 225-237.

[3] Melham, R.S., Some Analogs of the Identity $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$, Fibonacci Quarterly, (1999), 305-311.

[4] Natividad, L.R., On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, International Journal of Mathematics and Computing, 3(2) (2013), 38-40.

[5] B. Singh, P. Bhadouria, O. Sikhwal, K. Sisodiya, A Formula for Tetranacci-Like Sequence, Gen. Math. Notes, 20(2) (2014), 136-141.

[6] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[7] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 7(2), 45-56, 2019.

[8] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, Earthline Journal of Mathematical Sciences, 5(2), 297-327, 2021. https://doi.org/10.34198/ejms.5221.297327

[9] Soykan, Y., A Study On the Recurrence Properties of Generalized Tetranacci Sequence, International Journal of Mathematics Trends and Technology, 67(8), 185-192, 2021. doi:10.14445/22315373/IJMTT-V67I8P522

[10] Soykan, Y., A Study on Generalized Fibonacci Polynomials: Sum Formulas, International Journal of Advances in Applied Mathematics and Mechanics, 10(1), 39-118, 2022. (ISSN: 2347-2529)

[11] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, Earthline Journal of Mathematical Sciences, 11(1), 23-114, 2023. E-ISSN: 2581-8147. https://doi.org/10.34198/ejms.11123.23114

[12] Soykan, Y. On the Recurrence Properties of Generalized Tribonacci Sequence, Earthline Journal of Mathematical Sciences, 6(2), 253-269, 2021. https://doi.org/10.34198/ejms.6221.253269

[13] Soykan Y., On Generalized Narayana Numbers, Int. J. Adv. Appl. Math. and Mech. 7(3), 43-56, 2020.

[14] M. E. Waddill, Another Generalized Fibonacci Sequence, M. E., Fibonacci Quarterly, 5(3) (1967), 209-227.

[15] M. E. Waddill, The Tetranacci Sequence and Generalizations, Fibonacci Quarterly, (1992), 9-20.